NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2552

CONSIDERATIONS ON THE EFFECT OF WIND-TUNNEL WALLS ON OSCILLATING AIR FORCES FOR TWO-DIMENSIONAL SUBSONIC COMPRESSIBLE FLOW

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Page 6: In equations (9) and (10) the exponential term $e^{i\omega\left(t - \frac{V}{x}\right)}$ should be changed to $e^{i\omega\left(t - \frac{x}{V}\right)}$.

Page 7: In equation (13) the plus sign preceding the last term (the integral) should be changed to a minus sign.

Page 9: In equation (17) the summation of terms

$$\frac{2i\beta}{\pi}\sum_{n=1}^{\infty}(-1)^{n}e^{-\frac{\omega}{V}|y-nH|}\log\left(\frac{1 + \sqrt{1 - M^2}}{M}\right)$$

should be deleted and replaced by the summation

$$\frac{i\beta\omega}{V}\sum_{n=1}^{\infty}\left\{-\frac{V}{\pi\omega}\log\left(\frac{1 + \sqrt{1 - M^2}}{M}\right) + \frac{2i\beta^2 H}{M^2}\left[\frac{1}{\sqrt{\left(\frac{\pi\beta}{M}\right)^2 - \left(\frac{\omega H}{V}\right)^2}}\ \frac{1}{\sqrt{\left(\frac{\pi\beta}{M}\right)^2 - \left(\frac{\omega H}{V}\right)^2} + \frac{i\omega H}{V}} + \right.\right.$$

$$\left.\left.\frac{1}{\sqrt{(2n + 1)^2\left(\frac{\pi\beta}{M}\right)^2 - \left(\frac{\omega H}{V}\right)^2}}\ \frac{1}{\sqrt{(2n + 1)^2\left(\frac{\pi\beta}{M}\right)^2 - \left(\frac{\omega H}{V}\right)^2} + \frac{i\omega H}{MV}}\right]\right\}$$

This last item warrants the following explanation: The expression

$$\frac{2i\beta}{\pi}e^{-\frac{\omega}{V}|y-nH|}\log\left(\frac{1 + \sqrt{1 - M^2}}{M}\right)$$

was obtained by evaluating the integral

$$\int_0^{\infty} e^{-\frac{i\omega\xi}{V\beta^2}} H_0^{(2)}\left[\frac{\omega}{c\beta^2}\sqrt{\xi^2 + \beta^2(y - nH)^2}\right]d\xi$$

by comparison with the known value of the integral

$$\int_0^{\infty} e^{-ist} H_0^{(2)}\left(a\sqrt{t^2 - b^2}\right)dt = \frac{2}{\pi}\,\frac{e^{-ib\sqrt{s^2-a^2}}}{\sqrt{s^2 - a^2}}\log\left(\frac{s + \sqrt{s^2 + a^2}}{a}\right)$$

This procedure leads to an incorrect result since the branch points of the Hankel functions involved are on the real axis in one case and on the imaginary axis in the other.

The integral being considered actually appears in an infinite summation of integrals, namely

$$\sum_{n=1}^{\infty}(-1)^n\int_0^{\infty} e^{-\frac{i\omega\xi}{V\beta^2}} H_0^{(2)}\left(\frac{\omega}{c\beta^2}\sqrt{\xi^2 + \beta^2 n^2 H^2}\right)d\xi =$$

$$\int_0^{\infty} e^{-\frac{i\omega\xi}{V\beta^2}} \sum_{n=1}^{\infty}(-1)^n H_0^{(2)}\left(\frac{\omega}{c\beta^2}\sqrt{\xi^2 + \beta^2 n^2 H^2}\right)d\xi$$

A correct value of this summation is obtained by replacing the series of Hankel functions in the last integral by the equivalent series of exponential functions given in equation (18) of the report and then performing the indicated integrations.

This procedure leads to the following final expression for w_1 (eq. (17))

$$w_1 = \frac{2A\pi\omega}{V^2} e^{i\omega\left(t - \frac{x}{V}\right)} \sum_{n=1}^{\infty} \left(-e^{\frac{i\omega x}{V\beta^2}} (-1)^n \left(1 + \frac{iV\beta^2}{\omega} \frac{\partial}{\partial x}\right) H_0^{(2)} \left[\frac{\omega}{c\beta^2}\sqrt{x^2 + (\beta n H)^2}\right] + \right.$$

$$\frac{i\beta\omega}{V} \left\{ -\frac{V}{\pi\omega} \log\left(\frac{1 + \sqrt{1 - M^2}}{M}\right) + \frac{2i\beta^2 H}{M^2} \left[\frac{1}{\sqrt{\left(\frac{\pi\beta}{M}\right)^2 - \left(\frac{\omega H}{V}\right)^2}} \frac{1}{\sqrt{\left(\frac{\pi\beta}{M}\right)^2 - \left(\frac{\omega H}{V}\right)^2 + \frac{i\omega H}{MV}}} + \right.\right.$$

$$\left.\left. \frac{1}{\sqrt{(2n + 1)^2\left(\frac{\pi\beta}{M}\right) - \left(\frac{\omega H}{V}\right)^2}} \frac{1}{\sqrt{(2n + 1)^2\left(\frac{\pi\beta}{M}\right) - \left(\frac{\omega H}{V}\right)^2 + \frac{i\omega H}{MV}}} \right] \right\} +$$

$$\left. i\beta^2(-1)^n \int_0^{\frac{\omega x}{V\beta^2}} e^{iu} H_0^{(2)} \left[M\sqrt{u^2 + \left(\frac{\omega n H}{V\beta}\right)^2} du\right] \right)$$

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SUMMARY

This paper treats the effect of wind-tunnel walls on the oscillating two-dimensional air forces in a compressible medium. The walls are simulated by the usual method of placing images at appropriate distances above and below the wing. An important result shown is that, for certain conditions of wing frequency, tunnel height, and Mach number, the tunnel and wing may form a resonant system so that the forces on the wing are greatly changed from the condition of no tunnel walls. It is pointed out that similar conditions exist for three-dimensional flow in circular and rectangular tunnels and apparently, within certain Mach number ranges, in tunnels of nonuniform cross section or even in open tunnels or jets.

INTRODUCTION

The understanding of flutter and other nonsteady phenomena requires a knowledge of the associated unsteady flow. In the underlying theories of unsteady flow, such assumptions as small displacements, linearizations, and an inviscid fluid are made in order to obtain workable and usable results. When it is necessary to investigate the effect of these assumptions on analytical results by measurements of the forces and moments on an oscillating wing in a wind tunnel or to treat cases that do not conform to theory, the question of the effect of the tunnel walls naturally arises. In the case of steady flow the problem of the effect of tunnel walls is more or less classic and has been treated by many investigators. In general, these investigators have been able to obtain relatively simple factors which can be used to modify measurements of the air forces on a wing in a tunnel to correspond to free-air conditions. The extension of the results to compressible flow presents no difficulties since the results for incompressible flow can be corrected according to Prandtl-Glauert correction factors.

In the case of unsteady flow, Reissner, reference 1, and W. P. Jones, reference 2, have published papers showing the effect of wind-tunnel walls for the incompressible case. In both papers, the influence of the tunnel walls is found to be comparatively small for most cases, although indications are given that, for some ranges of a reduced-frequency parameter, the effect may be quite large. In the unsteady case, unlike the steady case, the transition from results for incompressible flow to those for compressible flow cannot be accomplished by simple transformations. This difficulty is a result of the fact that, in an incompressible fluid, the velocity of propagation of a disturbance is infinite and no time lag occurs between the initiation of a disturbance and its effect at another position in the field, but, in a compressible fluid, a definite time is required for a signal to reach a distant field point so that both a phase lag and a change in magnitude result. Under certain conditions this phase lag can result in a resonant condition which would involve large corrections.

The purpose of this paper is to consider the effect of wind-tunnel walls on the forces on an oscillating airfoil of infinite span with considerations of the compressibility of the fluid. The usual method of images is employed in order to satisfy the condition of no normal velocity at the tunnel walls. First, the effect of tunnel walls on the induced vertical velocity, hereinafter referred to as downwash, of an oscillating doublet is determined and this result is used to formulate the integral equation for the downwash of an oscillating airfoil in a tunnel. This paper is not intended to give numerical values or any detailed calculations of final tunnel-wall correction factors but mainly to show the existing need for such calculations and to present equations for calculating corrections for the two-dimensional case.

SYMBOLS

A	constant
b	semichord
c	velocity of sound
H	tunnel height
$H_0^{(2)}$, $H_1^{(2)}$	Hankel functions
M	Mach number
Δp	local pressure difference

t	time
$u = \frac{\omega\xi}{V\beta^2}$	
V	velocity
w, w_0, w_1	downwash or vertical induced velocity
$\xi, x, y, z,$	Cartesian coordinates
$\beta = \sqrt{1 - M^2}$	
γ	Euler's constant
ω	angular frequency
λ	wave length
ψ	acceleration potential
φ	velocity potential
ρ	fluid density

ANALYSIS

Effect of Tunnel Walls on the Downwash of a Single Doublet

The differential equation that governs flow due to small nonsteady perturbations imposed on a steady, uniform flow field is the wave equation. Referred to rectangular coordinates, fixed relative to the undisturbed stream at infinity, this equation is

$$(1 - M^2)\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} - \frac{2M}{c}\frac{\partial^2\psi}{\partial x\,\partial t} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0 \qquad (1)$$

In this equation the independent variable ψ may be regarded as either a perturbation velocity potential or as an acceleration potential. In treating the boundary conditions of the second section of this analysis it is convenient to regard ψ as an acceleration potential. Thus, in order to be consistent, ψ is hereinafter regarded as an acceleration potential. Accordingly ψ is directly proportional to a perturbation pressure field and is therefore related to a perturbation velocity potential φ as follows:

$$\psi = \frac{\partial\varphi}{\partial t} + V \frac{\partial\varphi}{\partial x} \tag{2}$$

In order to calculate the downwash $w = \frac{\partial\varphi}{\partial y}$ associated with ψ, it is necessary to solve equation (2) for φ in terms of ψ.

When ψ and φ are sinusoidal functions of time, such that

$$\left.\begin{aligned} \psi(x,y,t) &= e^{i\omega t}\bar{\psi}(x,y) \\ \varphi(x,y,t) &= e^{i\omega t}\bar{\varphi}(x,y) \end{aligned}\right\} \tag{3}$$

equation (2) becomes independent of time and thus reduces to an equation with one dependent variable, namely

$$\bar{\psi} = i\omega\bar{\varphi} + V \frac{\partial\bar{\varphi}}{\partial x} \tag{4}$$

This equation can be integrated with respect to x to give

$$\bar{\varphi} = \frac{e^{-\frac{i\omega}{V}x}}{V} \int_{-\infty}^{x} \bar{\psi}(\xi,y) e^{\frac{i\omega}{V}\xi} \, d\xi \tag{5}$$

where the lower limit of integration is chosen for later convenience so that φ vanishes far ahead of the point of disturbance. The downwash may be readily calculated with the use of this equation. In the absence of tunnel walls the retarded potential ψ_0 (that is, the potential corresponding to outgoing waves) of a harmonically pulsating pressure doublet located, for simplicity, at (0,0) that satisfies equation (1) is

$$\psi_0 = -A\beta e^{i\omega\left(t+\frac{Mx}{c\beta^2}\right)} \frac{\partial}{\partial y}\int_{-\infty}^{\infty} \frac{e^{-\frac{i\omega}{c\beta^2}\sqrt{x^2+\beta^2y^2+\beta^2z^2}}}{\sqrt{x^2+\beta^2y^2+\beta^2z^2}}\, dz$$

$$= A\pi i e^{i\omega\left(t+\frac{Mx}{c\beta^2}\right)} \frac{\partial}{\partial y} H_0^{(2)}\left(\frac{\omega}{c\beta^2}\sqrt{x^2+\beta^2y^2}\right)$$

$$= -\frac{i\omega A\pi}{c} e^{i\omega\left(t+\frac{Mx}{c\beta^2}\right)} \frac{y}{\sqrt{x^2+\beta^2y^2}} H_1^{(2)}\left(\frac{\omega}{c\beta^2}\sqrt{x^2+\beta^2y^2}\right) \qquad (6)$$

where $H_0^{(2)}$ and $H_1^{(2)}$ are Hankel functions as defined in reference 3, A is an arbitrary constant denoting doublet strength, ω is circular frequency, and $\beta = \sqrt{1-M^2}$. The Hankel function $H_1^{(2)}$ in equation (6) becomes infinite $\left(\text{as } \frac{1}{\sqrt{x^2+\beta^2y^2}}\right)$ as its argument approaches zero. Otherwise $H_1^{(2)}$ is continuous and approaches zero as its argument approaches infinity. Thus the only discontinuity in ψ_0 is at the location of the doublet, that is, at $(x=0,y=0)$.

In the presence of plane tunnel walls located parallel to the x-axis at $H/2$ units above and $H/2$ units below the doublet position, the potential ψ of a pressure doublet may be represented by the potential of an infinite system of appropriately chosen reflecting doublets, namely (see fig. 1)

$$\psi = A\pi i e^{i\omega\left(t+\frac{Mx}{c\beta^2}\right)} \frac{\partial}{\partial y} \sum_{n=-\infty}^{\infty} (-1)^n H_0^{(2)}\left[\frac{\omega}{c\beta^2}\sqrt{x^2+\beta^2(y-nH)^2}\right] \qquad (7)$$

In this equation the term corresponding to $n = 0$ is the potential ψ_0, equation (6), discussed in the preceding paragraph. It may be noted that only this term of the infinite summation in equation (7) gives rise to a discontinuity in ψ at any point within the tunnel $\left(-\frac{H}{2}\leq y\leq\frac{H}{2}, -\infty<x<\infty\right)$.

The infinity of terms corresponding to $n \neq 0$ are necessary to cause the downwash w to vanish at all points of the tunnel walls, $y = \pm\frac{H}{2}$.

The downwash along the midsection of the tunnel $y = 0$ is given by

$$w = \frac{A\pi i}{V} e^{i\omega\left(t - \frac{x}{V}\right)} \lim_{y\rightarrow 0} \int_{-\infty}^{x} \sum_{n=-\infty}^{\infty} (-1)^n e^{\frac{i\omega\xi}{V\beta^2}} \frac{\partial^2}{\partial y^2} H_0^{(2)}\left[\frac{\omega}{c\beta^2}\sqrt{\xi^2 + \beta^2(y - nH)^2}\right] d\xi$$

$$= w_0 + w_1 \tag{8}$$

where

$$w_0 = \frac{A\pi i}{V} e^{i\omega\left(t - \frac{V}{x}\right)} \lim_{y\rightarrow 0} \int_{-\infty}^{x} e^{\frac{i\omega\xi}{V\beta^2}} \frac{\partial^2}{\partial y^2} H_0^{(2)}\left(\frac{\omega}{c\beta^2}\sqrt{\xi^2 + \beta^2 y^2}\right) d\xi \tag{9}$$

represents the downwash associated with the pressure doublet in the absence of tunnel walls and

$$w_1 = \frac{2A\beta\pi i}{V} e^{i\omega\left(t - \frac{V}{x}\right)} \lim_{y\rightarrow 0} \int_{-\infty}^{x} \sum_{n=1}^{\infty} (-1) e^{\frac{i\omega\xi}{V\beta^2}} \frac{\partial^2}{\partial y^2} H_0^{(2)}\left[\frac{\omega}{c\beta^2}\sqrt{\xi^2 + \beta^2(y - nH)^2}\right] d\xi \tag{10}$$

represents the additional downwash due to the presence of tunnel walls. Thus the relative value of w_0 as compared with $w_0 + w_1$ is the main item of interest here.

The integrals appearing in equations (9) and (10) can be reduced to simpler form for evaluation but since the steps required to reduce one of the integrals are the same as required to reduce the other, only the integral appearing in equation (9) will be treated in detail. The reduced form of the other integral can then be obtained by simple comparison. The Hankel function in equation (9) satisfies the following identity:

$$\frac{\partial^2}{\partial y^2} H_0^{(2)}\left(\frac{\omega}{c\beta^2}\sqrt{\xi^2 + \beta^2 y^2}\right) \equiv -\beta^2 \frac{\partial^2}{\partial \xi^2} H_0^{(2)}\left(\frac{\omega}{c\beta^2}\sqrt{\xi^2 + \beta^2 y^2}\right) - \frac{\omega^2}{\beta^2 c^2} H_0^{(2)}\left(\frac{\omega}{c\beta^2}\sqrt{\xi^2 + \beta^2 y^2}\right) \tag{11}$$

Substituting this relation into equation (9) gives

$$w_0 = -\frac{A\pi i}{V} e^{i\omega\left(t - \frac{x}{V}\right)} \lim_{y \to 0} \left[\beta^2 \int_{-\infty}^{x} e^{\frac{i\omega}{V\beta^2}\xi} \frac{\partial^2}{\partial \xi^2} H_0^{(2)}\left(\frac{\omega}{c\beta^2}\sqrt{\xi^2 + \beta^2 y^2}\right) d\xi + \frac{\omega^2}{\beta^2 c^2} \int_{-\infty}^{x} e^{\frac{i\omega}{V\beta^2}\xi} H_0^{(2)}\left(\frac{\omega}{c\beta^2}\sqrt{\xi^2 + \beta^2 y^2}\right) d\xi\right] \tag{12}$$

In equation (12) the first integral can be integrated twice by parts to give for w_0

$$w_0 = -\frac{A\pi i}{V} \lim_{y \to 0} e^{i\omega\left(t - \frac{x}{V}\right)} \left[-\frac{\omega}{c} e^{\frac{i\omega x}{V\beta^2}} \frac{x}{\sqrt{x^2 + \beta^2 y^2}} H_1^{(2)}\left(\frac{\omega}{c\beta^2}\sqrt{x^2 + \beta^2 y^2}\right) - \frac{i\omega}{V} e^{\frac{i\omega x}{V\beta^2}} H_0^{(2)}\left(\frac{\omega}{c\beta^2}\sqrt{x^2 + \beta^2 y^2}\right) + \frac{\omega^2}{V^2} \int_{-\infty}^{x} e^{\frac{i\omega\xi}{V\beta^2}} H_0^{(2)}\left(\frac{\omega}{c\beta^2}\sqrt{\xi^2 + \beta^2 y^2}\right) d\xi\right] \tag{13}$$

By writing the integral in equation (13) as the sum of two integrals, namely

$$\int_{-\infty}^{x} = \int_{-\infty}^{0} + \int_{0}^{x} \tag{14}$$

and making a change of variable

$$\frac{\omega\xi}{V\beta^2} = u \tag{15}$$

the expression for w_0 may be further reduced to

$$w_0 = \frac{A\pi\omega}{V^2} \lim_{y \to 0} e^{i\omega\left(t - \frac{x}{V}\right)} \left\{ iMe^{\frac{i\omega x}{V\beta^2}} \frac{x}{\sqrt{x^2 + \beta^2 y^2}} H_1^{(2)}\left(\frac{\omega}{c\beta^2}\sqrt{x^2 + \beta^2 y^2}\right) - \right.$$

$$e^{\frac{i\omega x}{V\beta^2}} H_0^{(2)}\left(\frac{\omega}{c\beta^2}\sqrt{x^2 + \beta^2 y^2}\right) + \frac{2i\beta}{\pi} e^{-\frac{\omega|y|}{V}} \log\left(\frac{1 + \sqrt{1 - M^2}}{M}\right) +$$

$$\left. i\beta^2 \int_0^{\frac{\omega x}{V\beta^2}} e^{iu} H_0^{(2)}\left[M\sqrt{u^2 + \left(\frac{\omega y}{V\beta}\right)^2}\right] du \right\} \tag{16}$$

In the limit $y = 0$ the expression in braces in equation (16) reduces to the kernel of Possio's integral equation relating pressure and downwash for the oscillating airfoil in compressible flow. (This result checks the results for this expression given, for example, in reference 4.)

The value of the integrals in equation (10) may be similarily reduced to give

$$w_1 = \frac{2A\pi\omega}{V^2} e^{i\omega\left(t - \frac{x}{V}\right)} \lim_{y \to 0} \sum_{n=1}^{\infty} (-1)^n \left\{ \frac{iMx}{\sqrt{x^2 + \beta^2(y - nH)^2}} e^{\frac{i\omega x}{V\beta^2}} H_1^{(2)}\left[\frac{\omega}{c\beta^2}\sqrt{x^2 + \beta^2(y - nH)^2}\right] - \right.$$

$$e^{\frac{i\omega x}{V\beta^2}} H_0^{(2)}\left[\frac{\omega}{c\beta^2}\sqrt{x^2 + \beta^2(y - nH)^2}\right] + \frac{2i\beta}{\pi} e^{-\frac{\omega}{V}|y-nH|} \log\left(\frac{1 + \sqrt{1 - M^2}}{M}\right) +$$

$$\left. i\beta^2 \int_0^{\frac{\omega x}{V\beta^2}} e^{iu} H_0^{(2)}\left[M\sqrt{u^2 + \frac{\omega^2}{V^2\beta^2}(y - nH)^2}\right] du \right\}$$

$$= \frac{2A\pi\omega}{V^2} e^{i\omega\left(t - \frac{x}{V}\right)} \sum_{n=1}^{\infty} (-1)^n \left\{ -e^{\frac{i\omega x}{V\beta^2}}\left(1 + \frac{iV\beta^2}{\omega}\frac{\partial}{\partial x}\right) H_0^{(2)}\left[\frac{\omega}{c\beta^2}\sqrt{x^2 + (\beta nH)^2}\right] + \right.$$

$$\left. \frac{2i\beta}{\pi} e^{-\frac{nH\omega}{V}} \log\left(\frac{1 + \sqrt{1 + M^2}}{M}\right) + i\beta^2 \int_0^{\frac{\omega x}{V\beta^2}} e^{iu} H_0^{(2)}\left[M\sqrt{u^2 + \left(\frac{\omega nH}{V\beta}\right)^2}\right] du \right\} \qquad (17)$$

In general, this infinite-series representation of w_1, equation (17), converges to a finite value. However, for certain critical values of the frequency parameter $\omega H/V$, it is found that the value of w_1 becomes infinite. This fact can be readily made evident by use of relations given in reference 5 where it is shown that an infinite series of Hankel functions of the type appearing in equation (17) can be replaced by an equivalent series of exponential functions as follows:

$$\sum_{n=1}^{\infty}(-1)^n H_0^{(2)}\left[\frac{\omega}{c\beta^2}\sqrt{x^2+(\beta nH)^2}\right] = \sum_{n=1}^{\infty}(-1)^n H_0^{(2)}\left[\frac{M\omega H}{V\beta}\sqrt{n^2+\left(\frac{x}{\beta H}\right)^2}\right]$$

$$= -\frac{1}{2}H_0^{(2)}\left(\frac{Mx}{\beta^2 H}\frac{\omega H}{V}\right) + \frac{i\beta}{M}\frac{e^{-\frac{Mx}{\beta^2 H}\sqrt{\left(\frac{\pi\beta}{M}\right)^2-\left(\frac{\omega H}{V}\right)^2}}}{\sqrt{\left(\frac{\pi\beta}{M}\right)^2-\left(\frac{\omega H}{V}\right)^2}} +$$

$$\frac{i\beta}{M}\sum_{m=1}^{\infty}\left[\frac{e^{-\frac{Mx}{\beta^2 H}\sqrt{(2m+1)^2\left(\frac{\pi\beta}{M}\right)^2-\left(\frac{\omega H}{V}\right)^2}}}{\sqrt{(2m+1)^2\left(\frac{\pi\beta}{M}\right)^2-\left(\frac{\omega H}{V}\right)^2}} + \right.$$

$$\left.\frac{e^{-\frac{Mx}{\beta^2 H}\sqrt{(2m-1)^2\left(\frac{\pi\beta}{M}\right)^2-\left(\frac{\omega H}{V}\right)^2}}}{\sqrt{(2m-1)^2\left(\frac{\pi\beta}{M}\right)^2-\left(\frac{\omega H}{V}\right)^2}}\right] \qquad (18)$$

It may be seen that, if this relation is substituted into equation (17), the value of w_1 becomes infinite for all values of x when the frequency parameter $\omega H/V$ has any of the values given by

$$\frac{\omega H}{V} = (2m-1)\frac{\pi\beta}{M} \qquad (m = 1, 2, 3, \ldots) \qquad (19)$$

These critical values of the frequency parameter correspond to a condition of pure resonance in the tunnel which in the present case implies that a harmonic disturbance of any finite amplitude may lead to a downwash of infinite amplitude.

Of course these infinite values of w_1 would never be realized under practicable conditions because factors such as finite tunnel length, absorption through walls, fluid viscosity, and so forth that would give rise to damping would make pure resonance unobtainable; however, with damping present, resonant frequencies yielding values of $\omega H/V$ in the neighborhood of those given in equation (19) would exist and it is not likely that quantitative agreement or even possibly qualitative agreement between calculated and measured downwash (or forces) can be realized when the value of $\omega H/V$ is in the neighborhood of these critical values.

It is interesting to note that the effect of boundary conditions such as section geometry, tunnel-wall flexibility, and so forth is to change the value of the critical frequency but not to do away with the possibility of resonance. Also, by treatments similar to those employed herein, it can be shown that under idealized conditions resonance can occur in three-dimensional flow in both round and rectangular tunnels or apparently, within certain Mach number ranges, in tunnels of nonuniform cross section (expanding or contracting section) or even in open tunnels or jets.

The fundamental or smallest critical values of $\omega H/V$, corresponding to $m = 1$ in equation (19), are shown plotted as functions of Mach number M in figure 2. This figure indicates that there is no finite critical value of $\omega H/V$ for the conditions $M = 0$, $V \neq 0$, and $c = \infty$, which correspond to a flow of incompressible fluid in the tunnel. This result agrees with those found in references 1 and 2.

The frequency parameter

$$\frac{\omega H}{c} = (2m - 1)\pi\beta \qquad (m = 1, 2, 3, \ldots) \qquad (20)$$

which may be derived from equation (19) is shown plotted, for $m = 1$, as a function of Mach number in figure 3. Equations (20) and figure 3 show that finite values of the critical frequency exist for the conditions $M = 0$, $V = 0$, and $c \neq \infty$. These conditions correspond to a compressible fluid at zero velocity in the tunnel. For these conditions equations (20) and the corresponding wave lengths

$$\lambda = \frac{2\pi c}{\omega} = \frac{2H}{2m - 1} \qquad (m = 1, 2, 3, \ldots) \qquad (21)$$

agree, respectively, with results found in the literature for the characteristic frequencies and wave lengths associated with transverse acoustic vibrations in rectangular chambers when the location of the source of disturbance is excluded as a nodal point. See, for example, reference 6.

It may be of interest to note that equation (20) can be derived from the principle of standing waves as follows: The condition for resonance for the type of disturbance considered implies that the standing transverse waves have a maximum velocity at the midsection of the tunnel and zero velocity at the boundaries. A half-sine wave of wave length $\lambda = 2H$ or any odd divisor of this length, namely, $\lambda = \frac{2H}{2m - 1}$ satisfies this condition. If c is the velocity of sound in the medium and V the velocity of the medium, the velocity of propagation of a disturbance in a fixed plane perpendicular to the air flow is $\sqrt{c^2 - V^2}$. Since the frequency is given by the speed of propagation divided by the wave length there is obtained

$$f = \frac{(2m - 1)\sqrt{c^2 - V^2}}{2H} = \frac{\omega}{2\pi}$$

or

$$\frac{\omega H}{c} = \pi\beta(2m - 1)$$

Integral Equation for an Airfoil of Infinite Aspect Ratio Oscillating in a Wind Tunnel

In order to present equations from which tunnel-wall corrections for two-dimensional flow can be calculated, use is made of the foregoing analysis to derive the integral equation, relating downwash distributions and lift distributions, for the effect of tunnel walls on the lift distribution associated with a given downwash distribution.

The resultant pressure or local lift Δp associated with the acceleration potential of a single doublet located at $(x_0,0)$ with strength depending on streamwise position x_0 may be expressed simply as (compare with equation (6)):

$$\Delta p = -2\rho \lim_{y\to 0} \psi\left[x_0,(x - x_0),y,t\right]$$

$$= -2\rho\pi i \lim_{y\to 0} A(x_0)e^{i\omega\left(t+\frac{Mx}{c\beta^2}\right)} \frac{\partial}{\partial y} H_0^{(2)}\left[\frac{\omega}{c\beta^2}\sqrt{(x - x_0)^2 + \beta^2y^2}\right] \tag{22}$$

where $A(x_0)$ denotes local doublet strength or lift density. The downwash due to a distribution of such doublets between $x_0 = -b$ and $x_0 = b$ is

$$w(x,t) = \frac{-2\rho\pi i}{V} e^{i\omega t} \lim_{y\to 0} \int_{-b}^{b} A(x_0)e^{-\frac{i\omega}{V}(x-x_0)} dx_0 \int_{-\infty}^{x-x_0} e^{\frac{i\omega\xi}{V\beta^2}} \frac{\partial^2}{\partial y^2} H_0^{(2)}\left(\frac{\omega}{c\beta^2}\sqrt{\xi^2 + \beta^2y^2}\right)d\xi \tag{23}$$

For a given value of the lift density $A(x_0)$, this equation determines the downwash. For a given or prescribed expression of $w(x,t)$, the distribution of lift density must be determined. Thus, in this case, equation (23) is a form of Possio's integral equation relating downwash and pressure for an airfoil oscillating in compressible flow. In passing it may be well to point out that Possio's equation has not yet been solved in closed form but has been evaluated by different methods of approximation by several authors. Reference 4 gives a résumé of these methods of approximation.

For an airfoil inside a two-dimensional tunnel the relation between downwash and local lift becomes (compare with equation (8))

$$w(x,t) = \frac{-2\rho\pi i}{V} e^{i\omega t} \lim_{y\to 0} \int_{-b}^{b} A(x_0) e^{-\frac{i\omega}{V}(x-x_0)} dx_0 \left\{ \int_{-\infty}^{x-x_0} e^{\frac{i\omega\xi}{V\beta^2}} \frac{\partial^2}{\partial y^2} H_0^{(2)}\left(\frac{\omega}{c\beta^2}\sqrt{\xi^2 + \beta^2 y^2}\right) d\xi \, + \right.$$

$$\left. 2\int_{-\infty}^{x-x_0} \sum_{n=1}^{\infty} (-1)^n e^{\frac{i\omega\xi}{V\beta^2}} \frac{\partial^2}{\partial y^2} H_0^{(2)}\left[\frac{\omega}{c\beta^2}\sqrt{\xi^2 + \beta^2(y - nH)^2}\right] d\xi \right\} \tag{24}$$

For a given value of lift density $A(x_0)$, this equation determines the effect of tunnel walls on the corresponding downwash. For a given downwash distribution, the more pertinent effect of tunnel walls on the distribution of lift density is obtained by comparing the solution of equation (23) with the solution of equation (24). In either case the summation in the second integral in braces in equation (24) is the same summation that was found in the preceding section to have critical values of the frequency parameter $\omega H/V$ that cause the summation to become infinite. Consequently, evaluations of equation (24) for values of the frequency parameter in the neighborhood of these critical values would lead to the same resonant effects found in the treatment of a single doublet. Otherwise, for values of the frequency parameter not too near critical values, it is proposed that a fairly close approximation to solutions of equations (23) and (24) for effects of tunnel walls on lift density (or lift) will generally yield results from which tunnel-wall correction factors for two-dimensional flow can be obtained. Expressions from which correction factors for three-dimensional flow can be obtained may be similarly derived when the downwash of a three-dimensional pressure doublet is employed instead of the downwash of a two-dimensional pressure doublet.

It appears desirable to solve equations (23) and (24) by collocation or some other approximate method to obtain tunnel-wall corrections for some particular cases of prescribed downwash and to determine experimentally the range, if any, of frequency parameter in which quantitative results can be obtained for these cases.

CONCLUDING REMARKS

The important result shown is that, in a tunnel of infinite length containing a flowing fluid, a resonant condition involving a transverse oscillation of the fluid across the tunnel is possible and measured air forces at or near this condition of resonance might be greatly modified from those measured in free air. This resonant condition is a (simple) function of Mach number, tunnel height, and wing frequency and brings to attention a new type of tunnel-wall interference.

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National Advisory Committee for Aeronautics
Langley Field, Va., September 24, 1951

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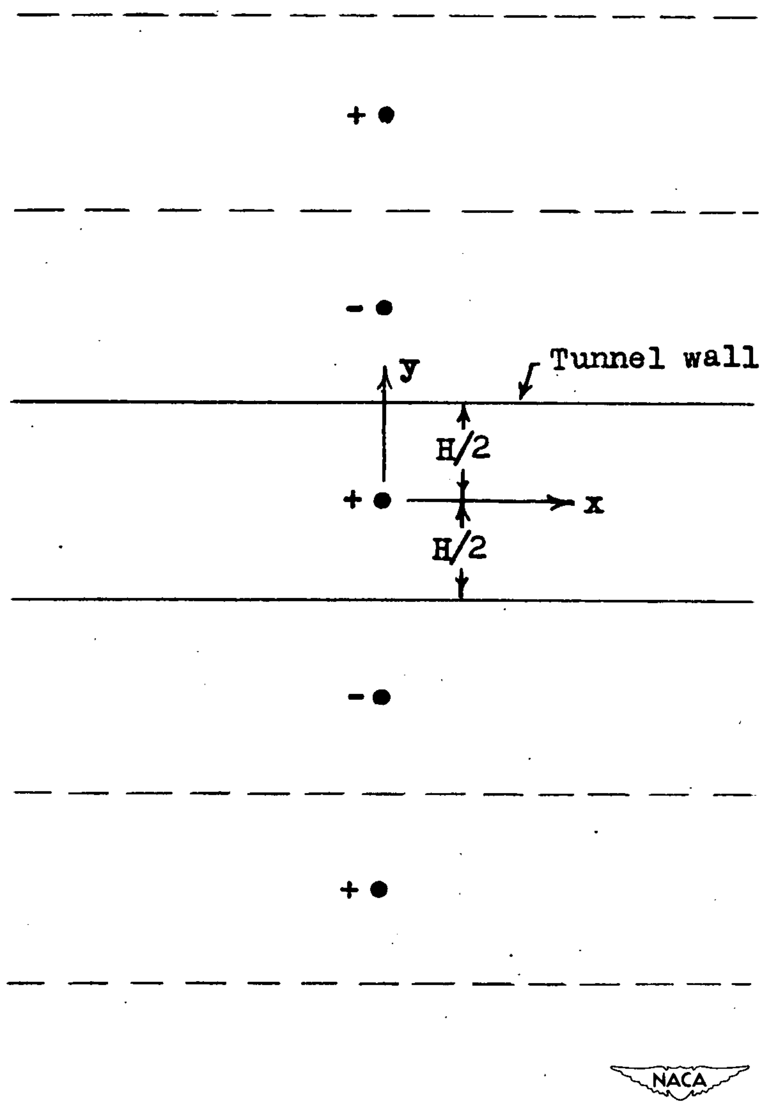

Figure 1.- Sketch showing reflecting system of doublets simulating two-dimensional tunnel walls.

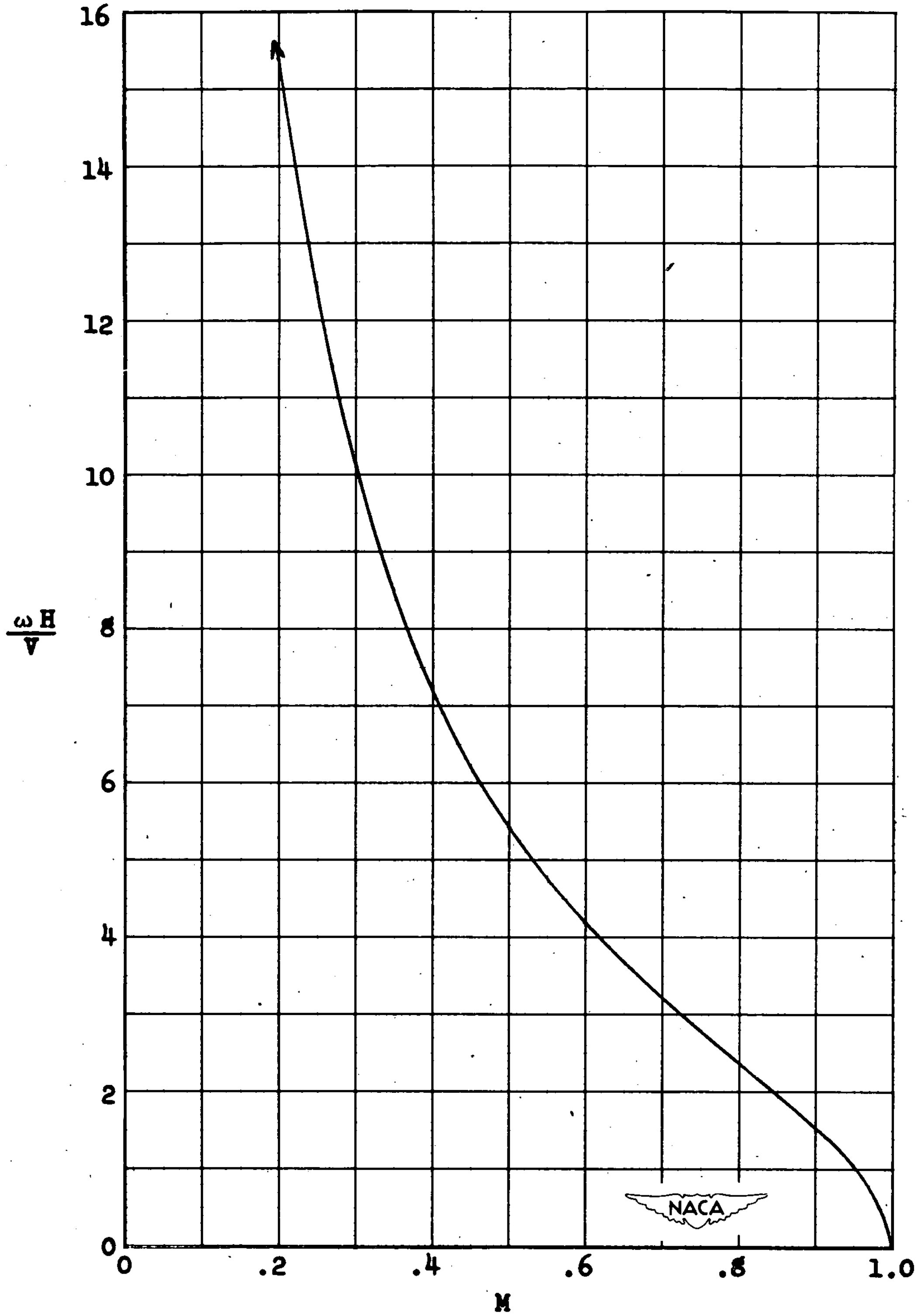

Figure 2.- Fundamental critical values of frequency parameter $\omega H/V$ plotted as a function of Mach number M.

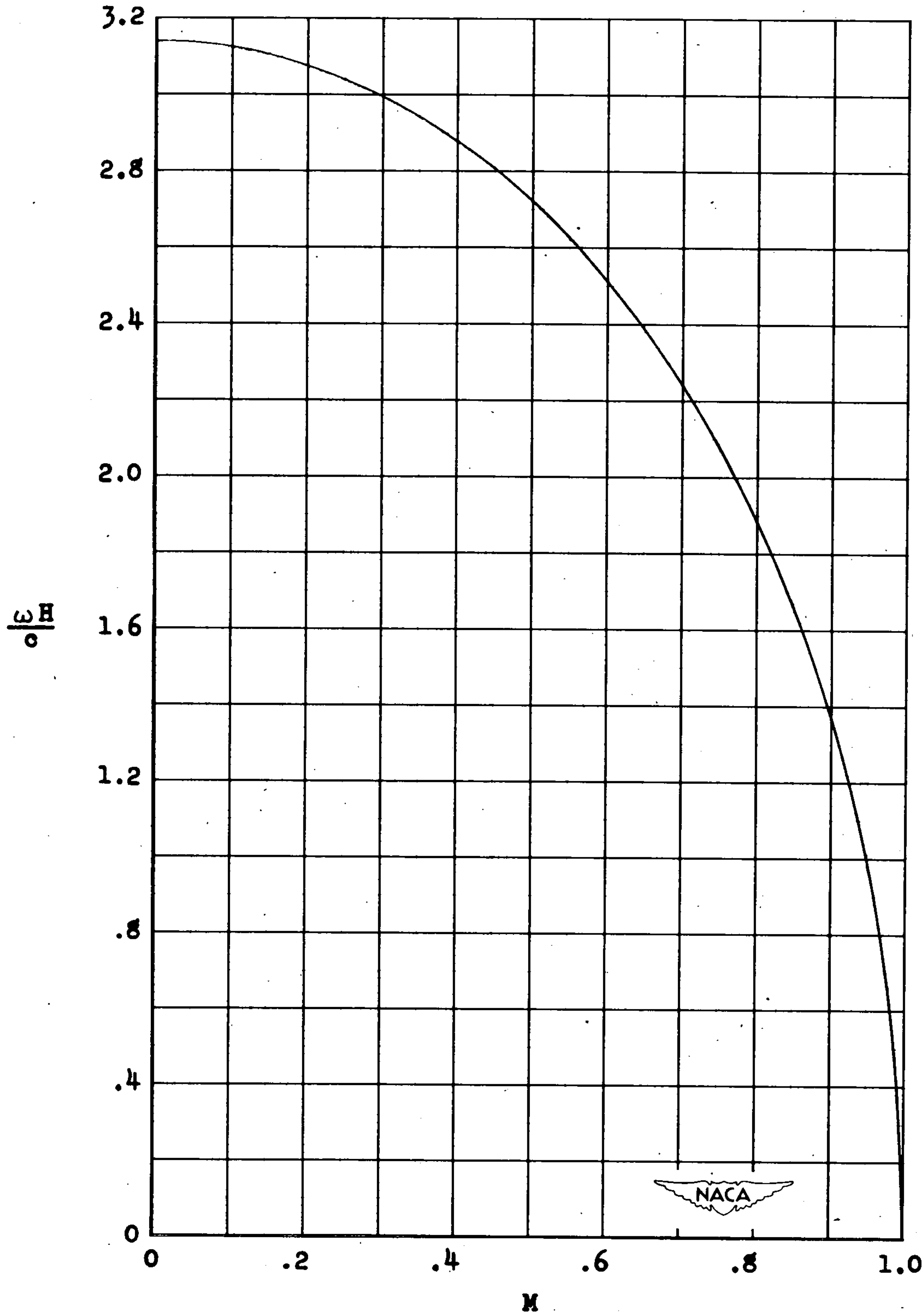

Figure 3.- Fundamental critical values of frequency parameter $\omega H/c$ plotted as a function of Mach number M.